Furuta's Inequality and Its Application to Ando's Theorem

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ABSTRACT

As a continuation of preceding notes, we discuss Furuta's inequality under the "chaotic order" defined by $\log A \geqslant \log B$ for positive invertible operators A and B, which is applied to a generalization of Ando's theorem. Consequently we obtain Furuta's inequality under the chaotic order.

1. INTRODUCTION

In [7], Furuta established an operator inequality which is an extension of the Löwner-Heinz inequality:

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FURUTA'S INEQUALITY. *Let A and B be positive operators acting on a Hilbert space. If $A \geqslant B \geqslant 0$, then*

$$(B^rA^pB^r)^{(1+2r)/(p+2r)} \geqslant B^{(1+2r)} \tag{1}$$

and

$$A^{(1+2r)} \geqslant (A^rB^pA^r)^{(1+2r)/(p+2r)} \tag{2}$$

for all $p \geqslant 1$ and $r \geqslant 0$.

Subsequently, Ando [1] proved an operator-monotone-like property for the exponential function. Inspired by this and the exponential order due to Hansen [10], we introduced the chaotic order $A \gg B$ by $\log A \geqslant \log B$ among positive invertible operators. It is just opposite to the exponential one. Thus Ando's result in [1] is rephrased:

THEOREM A. *For positive invertible operators A and B, the following conditions are equivalent:*

(a) $A \gg B$.
(b) *The following inequality holds for all $p \geqslant 0$:*

$$A^p \geqslant (A^{p/2}B^pA^{p/2})^{1/2}, \qquad i.e., \quad A^{-p}gB^p \leqslant I. \tag{3}$$

(c) *The operator function $G(p) = A^{-p}gB^p$ is monotone decreasing for $p \geqslant 0$, where g is the geometric mean.*

Now, we showed an operator inequality like Furuta's one in the preceding note [6] as follows: Let A and B be positive invertible operators. If $A \gg B$, then

$$(B^rA^pB^r)^{2r/(p+2r)} \geqslant B^{2r} \tag{4}$$

and

$$A^{2r} \geqslant (A^rB^pA^r)^{2r/(p+2r)} \tag{5}$$

for all $p \geqslant r \geqslant 0$. In this note, we first improve our preceding results (4) and (5); namely, they hold for all $p \geqslant 0$ and all $r \geqslant 0$. These inequalities are fundamental to showing monotonicity of an operator function discussed in [3], which is nothing else but an extension of Ando's theorem. As a consequence, we obtain Furuta's inequality under the chaotic order.

2. PRELIMINARY: MEANS OF OPERATORS

As in [4–6], the motif of our discussion is the means of operators introduced by Kubo and Ando [13]. A binary operation m among positive operators is called a mean if m is upper semicontinuous and satisfies

$$A \leqslant C \text{ and } B \leqslant D \quad \text{imply} \quad A\,m\,B \leqslant C\,m\,D$$

and the transformer inequality

$$T^*(A\,m\,B)T \leqslant T^*AT\,m\,T^*BT$$

for all T. We note that if T is invertible, then this inequality is replaced by the equality

$$T^*(A\,m\,B)T = T^*AT\,m\,T^*BT.$$

Now, by the principal result in [13], there is a unique mean m_s corresponding to the operator monotone function x^s for $0 \leqslant s \leqslant 1$:

$$1\,m_s\,x = x^s$$

for $t \geqslant 0$. In particular, the mean $g = m_{1/2}$ is called the geometric mean as in the case of scalars. In the sequel we write $m_{(p,s)} = m_{(1+s)/(p+s)}$ for all $p \geqslant 1$ and $s \geqslant 0$. Here we can state our recent result in [3], which is a nice application of Furuta's inequality.

THEOREM B. *If $A \geqslant B \geqslant 0$, then*

$$M(p,r) = B^{-2r}\,m_{(p,2r)}\,A^p \tag{6}$$

is a monotone increasing function, that is,

$$M(p+t, r+s) \geqslant M(p,r)$$

for $p \geqslant 1$ and $r, s, t \geqslant 0$.

On the other hand, we have attempted a mean-theoretic approach to Furuta's inequality in [2, 8, 11, 12]. The result is

$$M(p,r) = B^{-2r} m_{(p,2r)} A^p \geqslant B \tag{7}$$

and

$$N(p,r) = A^{-2r} m_{(p,2r)} B^p \leqslant A \tag{8}$$

under the assumption $A \geqslant B \geqslant 0$, $p \geqslant 1$, and $r \geqslant 0$. However, the argument in [12], [9], and [3] leads one to feel that the key point of Furuta's inequality should be

$$M(p,r) = B^{-2r} m_{(p,2r)} A^p \geqslant A \tag{9}$$

and

$$N(p,r) = A^{-2r} m_{(p,2r)} B^p \leqslant B \tag{10}$$

under the same assumption.

Concluding this section, we note that (4) and (5) can be rephrased to

$$B^{-2r} m_{2r/(p+2r)} A^p \geqslant 1 \tag{4'}$$

and

$$A^{-2r} m_{2r/(p+2r)} B^p \leqslant 1 \tag{5'}$$

respectively. If we take $p = 2r$ in (5'), then it is just (3) in Theorem A.

3. FURUTA'S INEQUALITY

In this section, we improve the Furuta-type inequalities (4) and (5). We need the following Löwner-Heinz inequality and the following known lemma.

THEOREM C. *$A \geqslant B \geqslant 0$ implies $A^s \geqslant B^s$ for $0 \leqslant s \leqslant 1$.*

LEMMA [9]. *Let A and B be invertible positive operators. For any real number r,*

$$(BAB)^r = BA^{1/2}(A^{1/2}B^2A^{1/2})^{r-1}A^{1/2}B.$$

Now we observe an operator inequality of Furuta's type, which is an improvement of our result (4) and (5) [6, Theorem 6].

THEOREM 1. *If $A \gg B$, then*

$$(B^r A^p B^r)^{2r/(p+2r)} \geqslant B^{2r} \tag{4}$$

and

$$A^{2r} \geqslant (A^r B^p A^r)^{2r/(p+2r)} \tag{5}$$

for all $p \geqslant 0$ and all $r \geqslant 0$.

Proof. First of all, by the Lemma we recall the following for any p $\geqslant 0$ and for any r $\geqslant 0$:

$$A^{2r} \geqslant (A^r B^p A^r)^{2r/(p+2r)} \quad \text{if and only if} \quad (B^{p/2} A^{2r} B^{p/2})^{p/(p+2r)} \geqslant B^p. \tag{$*$}$$

Put $C = A^p$ and $D = (A^{p/2} B^p A^{p/2})^{1/2}$. By Theorem A, if $A \gg B$, then $C \geqslant D \geqslant 0$ holds, so that Furuta's inequality (8) ensures

$$C \geqslant C^{-2t} m_{(a,2t)} D^a \qquad \text{for} \quad a \geqslant 1 \text{ and } t \geqslant 0,$$

that is,

$$A^{p(1+2t)} \geqslant \left\{ A^{pt} (A^{p/2} B^p A^{p/2})^{a/2} A^{pt} \right\}^{(1+2t)/(a+2t)} \qquad \text{for} \quad a \geqslant 1 \text{ and } t \geqslant 0$$

Put $a = 2$ in the above inequality:

$$A^{p(1+2t)} \geqslant (A^{p(t+1/2)} B^p A^{p(t+1/2)})^{(1+2t)/(2+2t)} \qquad \text{for} \quad t \geqslant 0.$$

Put $r = p(t + \frac{1}{2})$; then $(1+2t)/(2+2t) = 2r/(p+2r)$, so that

$$A^{2r} \geqslant (A^r B^p A^r)^{2r/(p+2r)} \qquad \text{for all } p \text{ and } r \text{ such that } 2r \geqslant p \geqslant 0, \tag{5_1}$$

since $r = p(t + \frac{1}{2}) \geqslant p/2$.

On the other hand, $A \gg B$ implies $C \geqslant D \geqslant 0$, and this is equivalent to the following inequality by ($*$):

$$(B^{p/2} A^p B^{p/2})^{1/2} \geqslant B^p \qquad \text{for all} \quad p \geqslant 0. \tag{5'}$$

Put $E = (B^{p/2}A^pB^{p/2})^{1/2}$ and $F = B^p$. Again applying Furuta's inequality (7) to (5′), then we have

$$F^{-2t}\, m_{(a,2t)}\, E^a \geqslant F \qquad \text{for} \quad a \geqslant 1 \text{ and } t \geqslant 0,$$

that is,

$$\left\{B^{pt}\left(B^{p/2}A^pB^{p/2}\right)^{a/2}B^{pt}\right\}^{(1+2t)/(a+2t)} \geqslant B^{p(1+2t)} \quad \text{for} \quad a \geqslant 1 \text{ and } t \geqslant 0.$$

Put $a = 2$ in the above inequality:

$$\left(B^{p(t+1/2)}A^pB^{p(t+1/2)}\right)^{(1+2t)/(2+2t)} \geqslant B^{p(1+2t)} \qquad \text{for} \quad t \geqslant 0.$$

Put $q = p(t + \frac{1}{2})$; then $(1 + 2t)/(2 + 2t) = 2q/(p + 2q)$, so that

$$\left(B^qA^pB^q\right)^{2q/(p+2q)} \geqslant B^{2q} \qquad \text{for} \quad 2q \geqslant p \geqslant 0,$$

since $q = p(t + \frac{1}{2}) \geqslant p/2$. This is equivalent to the following inequality by $(*)$:

$$A^p \geqslant \left(A^{p/2}B^{2q}A^{p/2}\right)^{p/(p+2q)} \qquad \text{for} \quad 2q \geqslant p \geqslant 0.$$

We interchange p with $2r$ and also $2q$ with p; then

$$A^{2r} \geqslant \left(A^rB^pA^r\right)^{2r/(p+2r)} \qquad \text{for all } p \text{ and } r \text{ such that } p \geqslant 2r \geqslant 0. \quad (5_2)$$

Hence we have (5) by (5_1) and (5_2); so the proof is complete, since (4) is equivalent to (5). ■

Theorem 1 and Theorem C imply the following corollary, which plays an important role in the next section:

COROLLARY 2. *If $A \gg B$, then for a given $r \geqslant 0$*

$$\left(B^rA^pB^r\right)^{s/(p+2r)} \geqslant B^s \tag{11}$$

for all $p \geqslant 0$ and $2r \geqslant s \geqslant 0$, and for a given $p \geqslant 0$

$$A^s \geqslant \left(A^{p/2}B^{2r}A^{p/2}\right)^{s/(p+2r)} \tag{12}$$

for all $r \geqslant 0$ and $p \geqslant s \geqslant 0$.

4. A GENERALIZATION OF ANDO'S THEOREM

Finally we discuss a generalization of Theorem A of Ando [1]. Such an attempt has been made in [3]; cf. also [9]. The purpose of this section is to continue it. As it turns out, it gives us an extension of Theorem B.

A modification of Theorem B might be considered as in [3]. Let us define

$$m_{(p,s,t)} = m_{(t+s)/(p+s)}$$

for $p \geqslant t \geqslant 0$ and $s \geqslant 0$. Clearly $m_{(p,s,1)} = m_{(p,s)}$.

THEOREM 3. *If $A \gg B$, then for a given $t \geqslant 0$*

$$M_t(p,r) = B^{-2r} m_{(p,2r,t)} A^p$$

is monotone increasing for $p \geqslant t$ and $r \geqslant 0$.

Proof. First of all, we prove that for a fixed $r > 0$, $M_t(p+s,r) \geqslant M_t(p,r)$ for $p \geqslant s \geqslant 0$. Putting $m = m_{(p+s,2r,t)}$, it follows from (12) that

$$\begin{aligned}
M_t(p+s,r) &= B^{-2r} m A^{p+s} \\
&= A^{p/2}(A^{-p/2}B^{-2r}A^{-p/2} m A^s) A^{p/2} \\
&\geqslant A^{p/2}\Big[(A^{p/2}B^{2r}A^{p/2})^{-1} m (A^{p/2}B^{2r}A^{p/2})^{s/(p+2r)}\Big] A^{p/2} \\
&= A^{p/2}(A^{-p/2}B^{-2r}A^{-p/2})^{(p-t)/(p+2r)} A^{p/2} \\
&= A^p m_{(p-t)/(p+2r)} B^{-2r} \\
&= B^{-2r} m_{(p,2r,t)} A^p.
\end{aligned}$$

The last equality is implied by the fact that $C m_a D = D m_{1-a} C$ for $1 \geqslant a \geqslant 0$.

Next we show the monotonicity in r. Putting $m = m_{(p,2r+s,t)}$ for $2r \geqslant s \geqslant 0$, it follows from (11) that

$$\begin{aligned}
M_t\left(p, r + \frac{s}{2}\right) &= B^{-r}(B^{-s} m B^r A^p B^r) B^{-r} \\
&\geqslant B^{-r}\Big[(B^r A^p B^r)^{-s/(p+2r)} m B^r A^p B^r\Big] B^{-r}
\end{aligned}$$

$$= B^{-r}(B^r A^p B^r)^{(t+2r)/(p+2r)} B^{-r}$$

$$= M_t(p, r).$$

■

As an immediate consequence of Theorem 3, Theorem A has the following generalization:

THEOREM 4. *For positive invertible operators A and B, the following conditions are equivalent*:

(a) $A \gg B$.
(b) *For each fixed* $t \geqslant 0$, $M_t(p, r) \geqslant A^t$ *for* $r \geqslant 0$ *and* $p \geqslant t$.
(c) *For each fixed* $t \geqslant 0$, $M_t(p, r)$ *is a monotone increasing function for* $r \geqslant 0$ *and* $p \geqslant t$.

Proof. (a) → (c) is already known by Theorem 3, and also (c) → (b) follows because $M_t(p, 0) = A^t$; so we have only to show (b) → (a), as follows. Put $t = 0$ and $2r = p$ in (b); then

$$(B^{p/2} A^p B^{p/2})^{1/2} \geqslant B^p.$$

This is equivalent to (b) in Theorem A, so we have (a) by Theorem A. ■

Finally, we mention that Furuta's inequality can be extended in the sense of (9). Namely, if we take $t = 1$ in (b) of Theorem 4, then we have:

COROLLARY 5. *If* $A \gg B$, *then*

$$M(p, r) = B^{-2r}\, m_{(p,2r)}\, A^p \geqslant A. \tag{9}$$

REMARK. We here note that Furuta's original inequality (7) does not hold under the weaker hypothesis $A \gg B$, in general. We can pose a simple example: Let

$$A = \begin{pmatrix} 29 & 16 \\ 16 & 13 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$A^{1/2} = \begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix} \geqslant \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = B^{1/2},$$

so that $A \not\geqslant B$ but $A \gg B$ by taking logarithm on both sides of $A^{1/2} \geqslant B^{1/2}$. Moreover it is easily checked that (7) for $p = 1 = 2r$ reduces to $A \geqslant B$. Hence (7) is false.

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